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Existence of Best Uniform Approximations by Reciprocals*

L. KEENER[†, ‡]

Department of Mathematics, Dalhousie University Halifax, Nova Scotia, Canada

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Let S be a nonempty set and define $\mathscr{M}$ to be the space of all bounded, real-valued functions on S. For $g \in \mathscr{M}$, define $\| g \| = \sup \{| g(x)| : x \in S\}$. If V is any finite-dimensional subspace, then every $f \in \mathscr{M}$ has a best approximation from V in the above norm. This is well known and easy to prove using a compactness argument [2]. The purpose of this paper is to develop an existence theory for approximation by reciprocals of functions in such a space V. While the complete generality of the above result is not achievable, existence can be demonstrated under relatively weak assumptions on f, S, and V. In particular, S need not be topologized. The arguments used are in part similar to arguments in the literature (see, for example, [1] or [3]).

In the last section we obtain two results in more specialized settings. The first is an extension of a known result and the second is a known result with a new and far shorter proof.

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Let $\mathscr{C}$ be a collection of subsets of S, closed under finite intersections. Call $\mu : \mathscr{C} \to [0, \infty]$ a zero monotone measure (zm-measure) if $A, B \in \mathscr{C}$, $\mu(B) = 0$, and $A \subset B$ implies $\mu(A) = 0$. Whenever we write $\mu(A)$ it will be implicitly understood that A is in the domain of μ.

DEFINITION. A function $f \in \mathscr{M}$ will be called sup measure continuous

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‡ Present address: Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada.

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(sm-continuous) with respect to μ if there is a sequence of subsets of S, $\langle B_j\rangle_{j=1}^{\infty}$ such that for some zm-measure μ,

(1) $\mu(B_j) > 0$ for all j,

(2) $x_j \in B_j$ for $j = 1, 2,\ldots$, implies $|f(x_j)| \to \|f\|$.

We then write $f \in \mathscr{M}(\mu)$.

DEFINITION. Let V be a subspace of $\mathscr{M}$ that is finite dimensional, contains the constant functions, and has the property that $\mu\{x \in S : p(x) = 0\} = 0$ for all $p \in V$, $p \not\equiv 0$, for some zm-measure μ. Then we say V is compatible with respect to μ. Also define $V_+^* = \{1/p : p \in V, p(x) > 0 \text{ for all } x \in S\}$.

LEMMA. *Let $\langle p_k\rangle$ be a sequence of functions in a space $V \subset \mathscr{M}$ that is compatible with respect to some zm-*measure μ and such that $\|p_k\| \to \infty$. *Then* there is a subsequence $\langle p_{k(i)}\rangle$ *such that* $|p_{k(i)}(x)| \to \infty$ *for all* $x \in S\backslash Z$, *where* Z *has* $\mu(Z) = 0$.

Proof. Define $\|p_k\| = M_k$ and assume without loss of generality that $M_k \neq 0$ for all k. Define $q_k = p_k/M_k$ for all k so $\|q_k\| = 1$ and $q_k \in V$. $\langle q_k\rangle$ is a bounded sequence in a finite-dimensional space, so there is a subsequence $\langle q_{k(i)}\rangle$ that converges to $q^* \in V$. Clearly $\|q^*\| = 1$ so $q^* \not\equiv 0$. Let $y \in S$ with $q^*(y) \neq 0$. (Note that we are taking $Z = q^{*-1}(0)$.) We must show that $|p_{k(i)}(y)| \to \infty$. Let $N > 0$ be given and choose I large enough to guarantee that $\|q_{k(i)} - q^*\| < |q^*(y)|/2$ and $\frac{1}{2}|q^*(y)| \cdot M_{k(i)} > N$ for $i \geqslant I$. For such i, $|q_{k(i)}(y) - q^*(y)| < |q^*(y)|/2$ implying $|q_{k(i)}(y)| > \frac{1}{2}|q^*(y)|$ and $|p_{k(i)}(y)| = M_{k(i)}|q_{k(i)}(y)| > |q^*(y)|/2 \cdot M_{k(i)} > N$.

THEOREM 1. *Let $f \in \mathscr{M}(\mu)$ and let $V \subset \mathscr{M}$ be compatible with respect to the same zm-measure μ. Define $z^* = \frac{1}{2}[\sup\{f(x): x \in S\} + \inf\{f(x): x \in S\}]$. Then f has a best uniform approximation from V_+^* if $z^* > 0$. If the functions of V_+^* are each bounded away from zero, then the converse is also true.*

Proof. Suppose $z^* > 0$ and let $\langle p_k\rangle$ be a sequence from V^* such that

$$\|f - 1/p_k\| \to \rho = \inf\{\|f - 1/p\|: p \in V, p > 0\}.$$

Then there is an $M > 0$ such that $\|p_k\| < M$. Indeed, suppose not. Without loss of generality we may assume $\|p_k\| \to \infty$. Let $L > 0$ be an integer for which $|f(x)| > \|f\| - z^*/3$ if $x \in B_L$, where B_L is as given in the definition of sm-continuous. Since $\mu(B_L) > 0$ we may use the lemma to show there is a $y \in B_L$ for which $|f(y)| > \|f\| - z^*/3$ and such that a subsequence of $\langle p_k\rangle$, denoted by $\langle p_{k(i)}\rangle$ can be found with $|p_{k(i)}(y)| \to \infty$. Then for sufficiently large i, $1/|p_{k(i)}(y)| < z^*/3$ and $|f(y) - 1/(p_{k(i)}(y))| > \rho + z^*/3$, since consideration of $z^* = 1/(1/z^*)$ shows that $\rho \leqslant \|f\| - z^*$. Therefore $\|f - (1/p_{k(i)})\| > \rho + (z^*/3)$ for sufficiently large i, contradicting the definition of ρ. Now there is a $\gamma > 0$, $\gamma < M$ such that for $p_k(x) < \gamma$, $1/(p_k(x)) \geqslant \|f\| + \rho + 1$. Let $Q = \{p \in V : \|p\| \leqslant M\}$ and $T = \{p \in V : p(x) \geqslant \gamma$ for all $x \in S\}$. Let $W = \{1/p : p \in Q \cap T\}$. Clearly a best approximation to f from W is a best approximation to f from V_+^*. But a best approximation to f from W is a function from the compact set $Q \cap T$ that minimizes the continuous functional $G(p) = \|f - 1/p\|$ and must therefore exist.

If $z^* \leqslant 0$ and if functions of V_+^* are bounded away from 0, consider the sequence $\langle 1/k \rangle$ for which $\|f - 1/k\| \to \rho$. Clearly there is no best approximation, and the converse of the theorem is true as well.

COROLLARY 1. *Let f be a function defined and bounded on S with the property that there is a sequence of distinct elements $x_i \in S$ such that $|f(x_i)| \to \|f\|$. Suppose the elements of V have at most a finite number of zeros or else vanish identically (e.g., algebraic polynomials). If $z^* > 0$, then f has a best uniform approximation from V_+^*.*

Proof. Define $\mu(A) = 0$ if A is finite and $\mu(A) = 1$ if A is infinite. Define $\langle B_j \rangle$ in the obvious fashion and apply Theorem 1.

COROLLARY 2. *Let f be a continuous function defined on $[a, b]$, a compact interval of real numbers. Suppose V is such that $p \in V$, $p \not\equiv 0$ implies $[a, b] \setminus p^{-1}(0)$ is dense in $[a, b]$. (This is called the dense nonzero property in* [4].) *If $z^* > 0$, then f has a best uniform approximation from V_+^*. (A generalization of this result is given in* [4].)

Proof. Define $\mu(A) = 0$ if the complement of A is dense in $[a, b]$ and define $\mu(A) = 1$ otherwise. Define $\langle B_j \rangle$ by $B_j = \{x \in [a, b] : f(x) \geqslant \|f\| - (1/j)\}$. Then $\mu(B_j) = 1$ for all j. Apply Theorem 1.

COROLLARY 3. *Let f be continuous and bounded on $(-\infty, \infty)$. Then if V is the space of trigonometric polynomials of degree $\leqslant n$, defined on $(-\infty, \infty)$, then f has a best uniform approximation from V_+^* if $z^* > 0$.*

Proof. Define $\mu(A) = 0$ if A is countable or finite, and define $\mu(A) = 1$ otherwise. Observe that all trigonometric polynomials have at most a countable number of zeros on $(-\infty, \infty)$ or vanish identically. Define $B_j = \{x \in (-\infty, \infty) : f(x) \geqslant \|f\| - 1/j\}$. Since f is continuous, $\mu(B_j) = 1$ for all j. Apply Theorem 1.

COROLLARY 4. *Let f be defined on the compact interval $[a, b]$. Partition $[a, b]$ into two Lebesgue measurable sets U_1 and U_2, with $\nu(U_1) > 0$ where ν represents Lebesgue measure. Define $\mu(A) = 0$ if $\nu(U_1 \cap A) = 0$ and define*

$\mu(A) = 1$ otherwise. Then if $f \in \mathscr{M}(\mu)$ and V is compatible with respect to μ, then f has a best uniform approximation from V_+^* if $z^* > 0$.

Proof. Immediate.

Remark. The above corollaries are examples of a large variety of statements that can be made as a consequence of Theorem 1. Note that the use of zero monotone measures is essentially a convenience.

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In this section we prove two results that do not follow immediately from the general theory. Theorem 2 is given for positive functions in [1]. Theorem 3 is given in [3].

THEOREM 2. *Let f be a continuous real-valued function, defined and bounded on $[a, \infty)$, where a is any real number. Suppose $z^* > 0$. If $V = \Pi_n$, the polynomials of degree n or less, then f has a best uniform approximation from V_*^+.*

Proof. Let $\langle p_k \rangle$ be a minimizing sequence, i.e., $\| f - 1/p_k \| \to \rho$. We may assume that for $x \geqslant K$, K sufficiently large that $f(x) \leqslant \| f \| - \epsilon$ for some $\epsilon > 0$. Indeed, if such a K cannot be found, the minimizing sequence must eventually consist of constants, since otherwise $\langle 1/p_k \rangle$ has a subsequence that converges to zero for arbitrarily large x, implying $\rho \geqslant \| f \|$ which is impossible since $z^* > 0$. We may then deduce that the function in V_+^* identically equal to z^* is the best approximation. Now suppose $\langle ||| p_k ||| \rangle$ is an unbounded sequence, where $||| \cdot |||$ is the sup norm on $[a, K]$. Assume without loss of generality that $||| p_k ||| \to \infty$. Then by the lemma, $p_k(x) \to 0$ except for a finite number of x in $[a, K]$. If μ is defined by $\mu(A) = 0$ if A is finite and $\mu(A) = 1$ otherwise, we see that a sequence $\langle B_j \rangle$ of sets in $[a, K]$ can be found such that $\mu(B_j) = 1$ and the B_j's satisfy the conditions for the sm-continuity of f restricted to $[a, K]$. We are then led to a contradiction as in Theorem 1. Clearly $p_k(x) \geqslant \gamma$ for all x in $[a, K]$ and some $\gamma > 0$. We may therefore deduce that a subsequence of $\langle p_k \rangle$, $\langle p_{k(i)} \rangle$, converges uniformly to a function $q \in V$, with $q > 0$ on $[a, K]$. Now $||| f - 1/p_{k(i)} ||| \leqslant \| f - 1/p_{k(i)} \| \to \rho$. Therefore by continuity, $||| f - 1/q ||| \leqslant \rho$, i.e., $\sup\{| f(x) - 1/q(x)| : x \in [a, K]\} \leqslant \rho$. If I is an interval, $[a, K] \subset I \subset [a, \infty)$, we may deduce that $p_{k(i)} \to q$ uniformly on I since the dimension of Π_n is $n + 1$ on any nontrivial interval. Therefore $\sup\{| f(x) - 1/q(x)| : x \in I\} \leqslant \rho$. Hence $\| f - 1/q \| = \rho$ and $1/q$ is a best approximation.

THEOREM 3. *Let $X \subset \mathbb{R}$ be compact, with* card $X \geqslant n + 2$ *and let f:*

$X \to \mathbb{R}$ be continuous and positive on X. Then there is a $p^* \in \Pi_n$, $p^* > 0$ such that

$$\| f - 1/p^* \| = \inf\{\| f - 1/p \| : p \in \Pi_n, p > 0 \text{ on } X\}.$$

Proof. Let $\langle 1/p_k \rangle$ be a minimizing sequence so that $\| f - 1/p_k \| \to \rho$, $p_k \in \Pi_n$, $p_k > 0$ for all k. By referring to the proof of Theorem 1, it is clear that existence will be established if we can show $\langle \| p_k \| \rangle$ is a bounded sequence. Suppose not, and without loss of generality assume $\| p_k \| \to \infty$. Define $q \in \Pi_n$, $q \geqslant 0$, $\| q \| = 1$ as in Theorem 1. Let $J = \{x_1, x_2, \ldots, x_k\}$, $k \leqslant n$ represent the zeros of q in X. (Initially, this set could be empty, but the proof that follows is still valid in this case.) Let $m = \sup\{f(x) : x \notin J\}$. Since card $X \geqslant n + 2$, m is well defined and positive. Let $J_1 = \{x \in X : f(x) > m\}$ and $J_2 = J \setminus J_1$. Clearly $J_1 \subset J$ and every point in J_1 is an isolated point of X. Indeed were $x \in J_1$ not isolated, continuity yields an infinite number of elements in J_1, a finite set. Assume without loss of generality that $J_1 = \{x_1, x_2, \ldots, x_p\}$ and $J_2 = \{x_{p+1}, x_{p+2}, \ldots, x_k\}$. By the argument of Theorem 1, $\rho \geqslant m$ since $1/p_k(x) \to 0$ for $x \notin J$. Let r be a polynomial in Π_n, not necessarily positive, such that

(1) $\dfrac{1}{r(x_i)} = f(x_i) \qquad i = 1, 2, \ldots, p$

and

(2) $\dfrac{1}{r(x_i)} = \dfrac{m}{2} \qquad i = p+1, p+2, \ldots, k.$

We may choose $L > 0$ so large that $r(x) + Lq(x) > 0$ for $x \in X$. Define $\langle 1/r_k \rangle$ by $r_k(x) = r(x) + kLq(x)$. Clearly $r_k(x) > 0$ for $x \in X$, $k = 1, 2, \ldots,$. Note that $r_k \in \Pi_n$. Now choose $\delta > 0$ sufficiently small so that $x \in (x_i - \delta, x_i + \delta) = D_i$ implies $| 1/(r(x)) - 1/(r(x_i))| \leqslant m/4$ for $i = p + 1, p + 2, \ldots, k$. Then for $x \in D_i \cap X$,

$$\frac{1}{r(x) + kLq(x)} \leqslant \frac{1}{r(x)} \leqslant \frac{1}{r(x_i)} + \frac{m}{4} = \frac{3m}{4}.$$

Let $F = X \setminus ((\bigcup_{i=p+1}^{k} D_i) \cup J_1)$. Since all the points in J_1 are isolated, F is compact and $\langle 1/r_k \rangle$ converges monotonically and pointwise to zero on F. By Dini's theorem the convergence is uniform so for a k sufficiently large we have

(1) $\dfrac{1}{r_k(x)} = f(x) \qquad \text{if } x \in J_1,$

(2) $\dfrac{1}{r_k(x)} \leqslant \dfrac{3m}{4} \qquad \text{if } x \in F,$

(3) $\dfrac{1}{r_k(x)} \leqslant \dfrac{3m}{4}$ if $x \in \bigcup_{i=p+1}^{k} D_i$,

(4) $\dfrac{1}{r_k(x)} \geqslant \epsilon$ on X for some sufficiently small $\epsilon > 0$.

Then $\| 1/r_k - f \| < m$, contradicting the observation that $\rho \geqslant m$. Hence $\langle \| p_k \| \rangle$ is bounded and existence is shown as in theorem 1.

We note that the condition $f > 0$ on X cannot be removed (see [5, p. 130]).

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